STRUCTURE OF INTERNAL SHIP WAVES IN A THREE-LAYER LIQUID WITH A STRATIFIED MIDDLE LAYER

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In determining the angle of the zone of wave disturbances of internal ship waves from the amplitude characteristics difficulties arise in finding the boundary of this zone [1, 2]. In this paper an attempt is made to fix the boundary of this region based on the lines of the crests and troughs of the wave disturbance in the wake behind a moving source.

In the case of surface ship waves the lines of constant phase have a turning point. In addition, one family of smooth curves corresponds to the long-wavelength part of the spectrum of disturbances (transverse waves) while another family corresponds to short-wavelength disturbances (longitudinal waves) [3]. The straight line passing through the turning point of the lines of constant phase makes with the axis of motion of the source an angle equal to one-half the Kelvin angle. The efficiency of excitation of one or another section of the spectrum of disturbances is determined by the ratio of the characteristic horizontal size of the source and the maximum length or the value of the standard Froude number.

Separate sections of the spectrum of internal ship waves in a three-layer liquid with a stratified middle layer were excited by varying the depth of a uniformly moving source. This corresponds to changing the characteristic horizontal size of the disturbance in the stratified layer above a moving source, which is valid for surface ship waves [4].

1. Theoretical Analysis. We shall write the equations for finding the vertical component of the velocity of the disturbances in a linear approximation, using the Boussinesq approximation, in dimensionless variables [3]:

$$\Delta w_1 = 0, \quad \Delta w_3 = 0, \quad \left(\frac{d^2}{dt^2}\Delta + \frac{1}{\mathrm{Fr}^2}\Delta_h\right) w_2 = 0,$$
$$\frac{d}{dt} = \frac{\partial}{\partial t} + \frac{\partial}{\partial x}, \quad \Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}, \quad \Delta_h = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{1.1}$$

Here the x axis is oriented along the incident flow, moving with a constant velocity U (U > 0) (the origin of the coordinate system is placed on the undisturbed bottom boundary of the stratified layer); the z axis is directed vertically upwards, opposite to the direction of gravity; w is the vertical component of the velocity vector; Fr = U/(NH) is the internal Froude number of the middle layer; and, H = const is the height of the undisturbed middle layer. We shall study a three-layer model of a liquid with a continuous density distribution and nonzero density gradient in the middle layer (N = const is the Brunt–Väisälä frequency of the middle layer). The height of the stratified layer H is taken as the linear scale, the ratio H/U is taken as the time scale, and the velocity of the incident flow U is taken as the velocity scale. All quantities referring to the top, middle, and bottom layers are denoted by the indices 1, 2, and 3, respectively.

The normal component of the velocity and the pressure must be continuous at the boundaries of the layers. For the vertical velocity we have the condition

$$w_1 = w_2, \quad \frac{\partial w_1}{\partial z} = \frac{\partial w_2}{\partial z} \quad \text{at} \quad z = 1;$$
$$w_2 = w_3, \quad \frac{\partial w_2}{\partial z} = \frac{\partial w_3}{\partial z} \quad \text{at} \quad z = 0, \tag{1.2}$$

requiring that the solutions in the top and bottom layers remain bounded in the limit $|z| \to \infty$.

We write out the dispersion relation for this problem:

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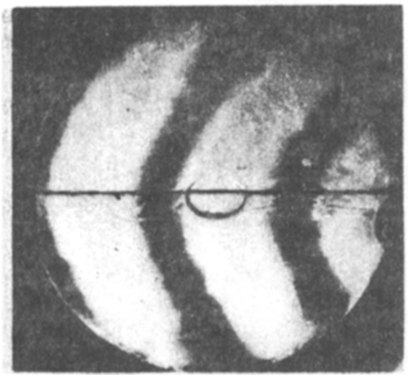

Fig. 1

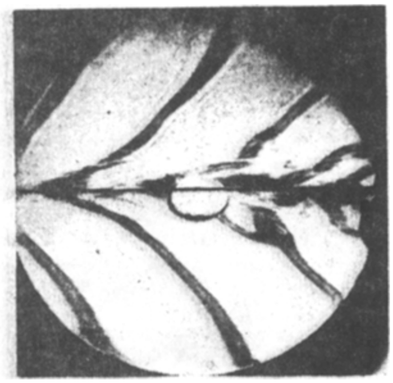

Fig. 2

$$\tan l = \frac{2kl}{l^2 - k^2}, \quad k = (k_1^2 + k_2^2)^{1/2}, \quad l^2 = k^2\left[\frac{1}{\mathrm{Fr}^2(\Omega - k_1)^2} - 1\right] \tag{1.3}$$

(k_1 and k_2 are the coordinates of the wave vector and Ω is the frequency [5, 6]). The dispersion relation consists of a denumerable number of branches $\Omega = \Omega_n$ (k_1, k_2, Fr), n = 0, 1, 2, ...

From (1.3) we can find the group velocity vector $\mathbf{V}_{\mathbf{gr}}^{(n)} = \left\{\frac{\partial \Omega_n}{\partial k_1}, \frac{\partial \Omega_n}{\partial k_2}\right\}$, which indicates the direction in which waves with frequency Ω and wave number **k** exist at large distances from the source of the disturbances for a wave system with number n. The lines of constant phase $\mathbf{kx}$ = A = const are given by the parametric expression $\mathbf{x}/A = \mathbf{V}_{\mathbf{gr}}^{(n)}/\mathbf{k}\mathbf{V}_{\mathbf{gr}}^{(n)}$. Here $\mathbf{x} = \{x, y\}$ and the wave vector **k** satisfies the dispersion relation for a fixed frequency Ω [3].

We shall write the components of the group-velocity vector in the stationary case (Ω = 0) in the form

$$V_{1\mathbf{gr}}^{(n)} = 1 + k_1 L, \quad V_{2\mathbf{gr}}^{(n)} = k_2 L, \qquad L = -k_1 l_n^2 / [k^2 (l_n^2 + k^2 + 2k)],$$

whence we find one-half the angle within which the wave disturbances are concentrated for the wave system with number n: $\theta_n = \max \arctan \left|\frac{V_{2\mathbf{gr}}^{(n)}}{V_{1\mathbf{gr}}^{(n)}}\right|$. The lines of constant phase for the zeroth mode (n = 0) as well as for wave systems with numbers n ($n \geq 1$) under the condition $n\pi \mathrm{Fr} < 1$ have a break and are qualitatively similar to the lines of constant phase of surface ship waves. At the point of the break of a line of constant phase the value of the wave vector corresponds to the point of inflection of the dispersion curve in the k_1, k_2 plane. For wave systems with number n, under the condition $n\pi \mathrm{Fr} > 1$, the dispersion curves do not contain points of inflection and the lines of constant phase are curves convex downwards and a sloping asymptote.

2. Experimental Results. The experimental studies of the phase structure of internal waves in a three-layer model were performed in a 1.0 × 0.35 × 0.25 m laboratory basin. The linear stratification of the middle layer of liquid was obtained by continuously injecting into the basin a water solution of sodium chloride with variable concentration. The densities of the uniform layers of liquid equalled the densities of the liquid at the top and bottom boundaries of the stratified layer.

The stratified layer was 4-5 cm high and the uniform layers were 6-7 cm high. Preliminary experiments established that increasing the vertical size of the uniform layers up to 15 cm has virtually no effect on the observed phase patterns. The experiments were performed immediately after the basin was filled, which made it possible to neglect the effect of the diffusion of salt on the boundaries of the stratified layer. The vertical distribution of the density of the liquid was measured with the help of an electric conductivity meter. A more accurate value of the Brunt–Väisälä frequency for the middle layer was determined based on the density distribution obtained [6]. The working values of the buoyancy frequencies equalled 1.3-1.5 $\sec^{-1}$.

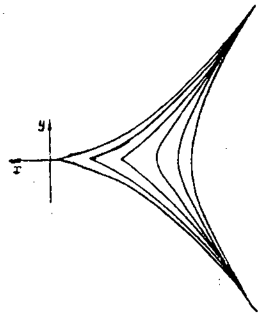

Fig. 3

Internal waves were excited in the stratified layer by means of uniform horizontal motion of a sphere 3.8 cm in diameter in the uniform bottom layer of the liquid (the velocities ranged from 1 to 6 cm/sec).

The motion of the source outside the stratified layer (or partially submerged in it) made it possible to excite efficiently the zeroth-order mode [6]. In this case the entire layer, as a whole, undergoes wave-like motion and the phase patterns in any horizontal planes are identical. The integral phase pattern, obtained with vertical transillumination of all layers of the liquid, will correspond to the real pattern of the lines of constant phase.

The phase patterns of the wave disturbances were visualized by the dark-field method using an IAB-451 shadow apparatus. A flat mirror, positioned at an angle of 45° to the vertical, was employed for vertical propagation of the light beam. The horizontal light beam from the illuminating part of the shadow apparatus passed, after being reflected from the flat mirror, through an optical window in the bottom of the basin, the layer of working liquid, and once again through an optical window, and after being reflected from a flat mirror at the top once again became horizontal and entered the receiving part of the shadow apparatus. The placement of the optical window at the top was determined by the need to eliminate random surface wave disturbances. To this end the bottom surface of the window was placed flush against the surface of the water. The optical windows were 20 cm in diameter. This dimension determined the region of simultaneous visualization. To reconstruct the complete wave pattern behind the sphere at distances of up to 15 units (in our case approximately 60 cm) it was necessary to record separate fragments of the wave pattern on film for each pass successively. The complete pattern was then reconstructed from separate, enlarged images from the negatives. In making the photographs the lines of zero gradient of the optical path (in the transilluminated layer of the liquid) along the direction of motion of the sphere (x axis) were recorded. These lines virtually coincide with the lines passing along the crests and troughs of the phase surface. For monitoring purposes the lines of zero gradient along the perpendicular direction (along the y axis) were also recorded. Within the limits of error of the measurements both lines coincide in the far zone.

Theoretical analysis established that the phase pattern of the internal waves in a stratified layer for the zeroth mode is structurally virtually identical to the phase pattern of surface ship waves. In both cases the wave disturbance in the wake behind the source consists of a continuous spectrum of plane waves, which has a long-wavelength limit. The direction of the wave vector of the wave with the minimum wave number coincides with the velocity vector of the source, and as the wave number increases the angle between these vectors increases and approaches $\pi/2$. Excitation of one or another part of the spectrum is

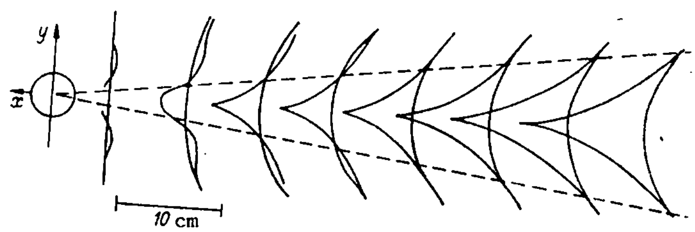

Fig. 4

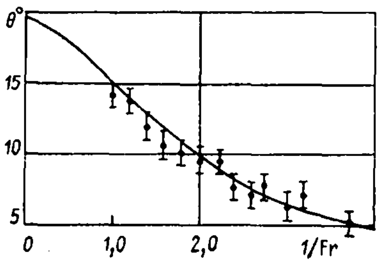

Fig. 5

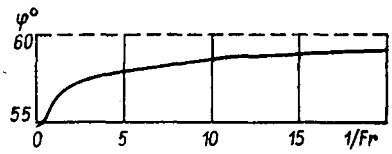

Fig. 6

determined by the ratio of the maximum wavelength and the size of the source. The dimensions of the laboratory tank precluded the use of large sources, so that the idea of changing the effective size of the source by submerging a small sphere (3.8 cm in diameter) at different depths under the stratified layer (as well as above it) was checked first. To this end the dependence of the phase pattern of the internal waves, arising with horizontal motion of a small sphere with fixed velocity, on the depth of submersion was studied.

Figure 1 shows a photograph of the phase pattern for motion of a small sphere with a velocity of 2.6 cm/sec at a depth of 5 cm (from the bottom boundary of the stratified layer up to the center of the sphere). Like all patterns of a given series, this fragment was obtained at a distance of 40 cm behind the source with H = 5 cm and N = 1.5 sec^{-1}. The phase pattern corresponds to the long-wavelength part of the spectrum of internal waves. The case of excitation of the short-wavelength part of the spectrum is illustrated in Fig. 2, when the source moves at zero depth. The distortions in the wake are caused by a vortex street. The technological part for fastening one of the flat mirrors can be seen at the center of both photographs. The entire family of phase patterns of a given series is shown in Fig. 3. The rightmost line corresponds to a submersion depth of 5 cm, and each subsequent line corresponds to a decrease in the depth by 1 cm. The obtained pattern of efficiency of excitation of different sections of the spectrum on internal waves versus the submersion depth of a source in the form of a sphere is not universal, and is determined by Fr = U/NH. As Fr is increased the value of the maximum wavelength in the given spectrum increases (the spectrum becomes broader) and to achieve efficient excitation of the long-wavelength part of the spectrum either the size of the source or its depth must be increased. The source size and depth indicated above made it possible to cover the working range Fr = 0.25-1.0.

The main part of the work consisted of finding a wedge (of the type of a wedge of surface Kelvin ship waves) for the zeroth mode in the stratified layer and measuring its half-angle as a function of Fr. To this end, the patterns of lines of crests and troughs were photographed for fixed values of Fr with two passes of the source (with different submersion depths). For one pass the pattern of wave disturbances of the long-wavelength part of the spectrum was reconstructed and the short-wavelength part of the spectrum was reconstructed with another pass. Then they were combined and the complete pattern of internal waves in the stratified layer was obtained. The typical complete pattern of lines of crests and troughs for Fr = 0.385 is presented in Fig. 4. The broken lines show the boundaries of the wedge; the half-angle of the wedge in this case equals 7 ± 1°. The configurations of crests and troughs in the far zone, where the finiteness of the source no longer has any effect, have turning points on the boundary of the wedge. To increase the accuracy of the measurement of the half-angles of the wedge an even larger number of turning points in the far zone must be recorded, and this requires a large laboratory basin. All experimental values obtained for the half-angles θ of the wedge are plotted on graphs of θ versus the inverse internal Froude number (1/Fr) in Fig. 5 as separate points (the line is analogous to the theoretical curve [5]). One can see that the experimental results agree well with the results of the linear theory.

The theoretical and experimental studies performed showed that the phase structure of the internal waves in a stratified layer for the zeroth mode is analogous to the phase pattern of surface Kelvin ship waves. In addition, the difference in the dispersion relations leads only to a difference in the dependence of the angle θ on the conditions of motion of the source. In the case of surface ship waves on deep water the half-angle of the wedge $\theta_w = \arcsin(1/3)$ does not depend on the velocity of the source, while for internal ship waves in a stratified layer it depends on Fr: as $Fr \to 0$ it approaches zero, while as $Fr \to \infty$ it approaches θ_W.

An even smaller difference is observed in the behavior of waves at the boundary of the wedge itself. The crests at the boundary of the wedge of surface ship waves make an angle of $\varphi_W = \arctan\sqrt{2}$ with the x axis. In the case of internal ship waves as $Fr \to \infty$ it approaches φ_W, and as Fr decreases it slowly grows and approaches 60°. Figure 6 shows the theoretical dependence of the slope angle of the tangent at the turning point of the line of constant phase with respect to the x axis on 1/Fr.

For small values of Fr the experimentally observed lines of constant phase are practically parallel straight lines. In addition, they pass outside the boundary of the wave zone, determined by the value of θ ($\theta \to 0$).

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SOME CLASSES OF TWO-DIMENSIONAL VORTEX FLOWS OF AN IDEAL FLUID

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There are relatively few known exact steady solutions of the two-dimensional Euler equations [1-3]. This is in part explained by the fact that the symmetry group of these equations is low [4]. But progress achieved in the study of nonlinear wave equations [5, 6] can be partially carried over to the study of elliptic problems. The purpose of the present paper is to obtain solutions of the equation for the stream function and to analyze these solutions. The solutions found here describe motion of the source type in a rotating fluid, periodic flow between two walls, motion in a rectangular cylinder, and others.

1. The stream function ψ for the two-dimensional steady flow of an ideal fluid satisfies the equation

$$\Delta\psi(x, y) = \omega, \tag{1.1}$$

where the vorticity ω is a function of ψ. For certain forms of the right hand side (1.1) can be solved using a modified separation of variables method and the Beklund transformation [6].

We assume that the vorticity is given by $\omega(\psi) = \varepsilon \sin \psi$ ($\varepsilon = \pm 1$). We look for a solution of (1.1) in the form [5] $\psi(x, y) = 4 \arctan(f)x)g(y))$, where the functions f and g satisfy the ordinary differential equations

$$f'^2 = nf^4 + mf^2 + k,\ g'^2 = kg^4 + (\varepsilon - m)g^2 + n \tag{1.2}$$

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